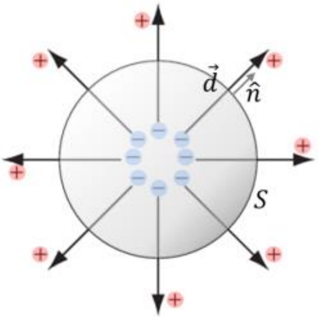
Chapter 12

Electromagnetic Waves in Matter

12-1 Maxwell's Equations in Matter

A. POLARIZATION CURRENT

(1) Consider a surface $\mathcal{S}$ bounding a volume $\mathcal{V}$ of a nonpolar dielectric. The application of an external field causes the bound charges: The positive charges flow out of $\mathcal{V}$ and the negative charges remain within the volume $\mathcal{V}$.



The charge crosses the surface $\mathcal{S}$ is

$$dQ_P = Nq\vec{d}\cdot\hat{n}da = \vec{P}\cdot\hat{n}da$$

The net charge through the surface is

$$Q_P = \oint_{\mathcal{S}} \vec{P}\cdot\hat{n}da$$

NOTE:

The bound charge satisfies the charge conservation law: we started with an electrically neutral dielectric body, the total charge of the body after polarization must remain zero.

The net charge within the volume is

$$Q_P = -\oint_{\mathcal{S}} \vec{P}\cdot\hat{n}da = \int_{\mathcal{V}} \left(-\nabla\cdot\vec{P}\right)d\tau \text{ and } Q_P = \int_{\mathcal{S}} \rho_b\, d\tau$$

Thus, we obtain

$$\rho_b = -\nabla\cdot\vec{P}$$

The total charge is

$$\oint_{\mathcal{S}} \sigma_b\, da + \int_{\mathcal{V}} \rho_b\, d\tau = \oint_{\mathcal{S}} \frac{dQ_p}{da} da - \int_{\mathcal{V}} \left(\nabla\cdot\vec{P}\right)d\tau$$
$$= \oint_{\mathcal{S}} \vec{P}\cdot\hat{n}\,da - \oint_{\mathcal{S}} \vec{P}\cdot\hat{n}\,da$$
$$= 0$$

(2) The charge flows through the surface per unit time is

$$I_P = \frac{dQ_P}{dt} = \oint_S \frac{\partial \vec{P}}{\partial t} \cdot \hat{n} da \text{ and } I_P = \int_S \vec{J}_P \cdot d\vec{a}$$

Thus, we obtain

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t} \cdots\cdots \text{polarization current}$$

NOTE:

The polarization current satisfy the continuity equation:

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \vec{J}_P = \frac{\partial \left(-\nabla \cdot \vec{P}\right)}{\partial t} + \nabla \cdot \frac{\partial \vec{P}}{\partial t} = -\nabla \cdot \frac{\partial \vec{P}}{\partial t} + \nabla \cdot \frac{\partial \vec{P}}{\partial t} = 0$$

B. MAXWELL'S EQUATIONS IN MATTER

(1) For fields in the presence of electric charge of density ρ and electric current, that is, charge in motion, of density $\vec{J}$. We have

$$① = \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

$$② = \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$③ = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \cdots\cdots \text{Gauss's law}$$

$$④ = \nabla \cdot \vec{B} = 0$$

(2) The electric charge density can be separated into two parts

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \vec{P}$$

The current density can be separated into three parts

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P = \vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Gauss's law (equation ③) can now be written as

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\left(\rho_f - \nabla \cdot \vec{P}\right) \Rightarrow \nabla \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \rho_f \Rightarrow \nabla \cdot \vec{D} = \rho_f$$

Meanwhile equation ② becomes

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}\right) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \nabla \times \left(\vec{B} - \mu_0 \vec{M}\right) = \mu_0 \vec{J}_f + \mu_0 \frac{\partial}{\partial t}\left(\vec{P} + \epsilon_o \vec{E}\right)$$

$$\Rightarrow \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

where $\vec{D}$ is called the electric displacement and $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is called the displacement current.

(3) Now Maxwell's equations, in terms of free charge and current, read

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho_f \cdots\cdots \text{Gauss's law}$$

$$\nabla \cdot \vec{B} = 0$$

For linear media, we have

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \text{ and } \vec{M} = \chi_m \vec{H}$$

which gives that

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu_0} - \chi_m \vec{H} \Rightarrow \vec{H} = \frac{\vec{B}}{\mu_0 (1 + \chi_m)} = \frac{\vec{B}}{\mu}$$

C. BOUNDARY-VALUE PROBLEMS WITH DIELECTRICS

(1) Maxwell's equations in integral form are

$$① = \oint_C \vec{E} \cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \cdots\cdots \text{Faraday's law}$$

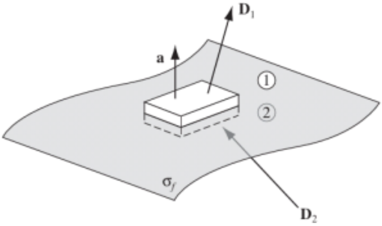
$$② = \oint_C \vec{H} \cdot d\vec{s} = I_f + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{a}$$

$$③ = \oint_S \vec{D} \cdot d\vec{a} = q_f \cdots\cdots \text{Gauss's law}$$

$$④ = \oint_S \vec{B} \cdot d\vec{a} = 0$$

where $\vec{E}$, $\vec{B}$, $\vec{D}$, and $\vec{H}$ will be discontinuous at a boundary between two different media, or at a surface that carries a charge density σ or a current density $\vec{K}$.

(2) We choose a Gaussian surface for a very tiny area $d\vec{a}$ and let the thickness go to zero.

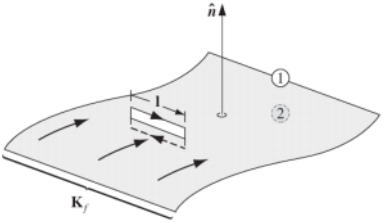


Thus, from equations ③ and ④, we obtain

$$\oint_S \vec{D}\cdot d\vec{a} = \underbrace{\vec{D}_1\cdot d\vec{a}}_{\text{media ①}} - \underbrace{\vec{D}_2\cdot d\vec{a}}_{\text{media ②}} = \sigma_f da \Rightarrow D_{1\perp} - D_{2\perp} = \sigma_f$$

$$\oint_S \vec{B}\cdot d\vec{a} = \underbrace{\vec{B}_1\cdot d\vec{a}}_{\text{media ①}} - \underbrace{\vec{B}_2\cdot d\vec{a}}_{\text{media ②}} = 0 \Rightarrow B_{1\perp} - B_{2\perp} = 0$$

We can choose a closed loop such that the width goes to zero as



Thus, we obtain

$$\oint_C \vec{E}\cdot d\vec{s} = \underbrace{\vec{E}_1\cdot d\vec{s}}_{\text{media ①}} - \underbrace{\vec{E}_2\cdot d\vec{s}}_{\text{media ②}} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} = 0 \Rightarrow E_{1\parallel} - E_{2\parallel} = 0$$

$$\oint_C \vec{H}\cdot d\vec{s} = \underbrace{\vec{H}_1\cdot d\vec{s}}_{\text{media ①}} - \underbrace{\vec{H}_2\cdot d\vec{s}}_{\text{media ②}} = I_f = \vec{K}_f\cdot(\hat{n}\times d\vec{s}) = \left(\vec{K}_f\times\hat{n}\right)\cdot d\vec{s}$$

$\Rightarrow H_{1\parallel} - H_{2\parallel} = \vec{K}_f \times \hat{n}$

For linear media, we have the boundary conditions:

$$\epsilon_1 E_{1\perp} - \epsilon_2 E_{2\perp} = \sigma_f$$
$$B_{1\perp} - B_{2\perp} = 0$$
$$E_{1\parallel} - E_{2\parallel} = 0$$
$$\frac{B_{1\parallel}}{\mu_1} - \frac{B_{2\parallel}}{\mu_2} = \vec{K}_f \times \hat{n}$$

12-2 Electromagnetic Waves in Matter

A. ELECTROMAGNETIC WAVES IN LINEAR MEDIA

(1) Inside matter, but in regions where there is no free charge or free current, Maxwell's equations become

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$
$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \cdot \vec{D} = 0 \cdots\cdots \text{Gauss's law}$$
$$\nabla \cdot \vec{B} = 0$$

If the medium is linear and homogeneous, i.e.,

$$\vec{D} = \epsilon \vec{E} \text{ and } \vec{H} = \frac{1}{\mu}\vec{B}$$

we have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$
$$\nabla \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$
$$\nabla \cdot \vec{E} = 0 \cdots\cdots \text{Gauss's law}$$
$$\nabla \cdot \vec{B} = 0$$

(2) Thus, electromagnetic waves propagate through a linear homogeneous medium at a speed v,

$$\nabla^2 \vec{E} = \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla^2 \vec{B} = \mu\epsilon\frac{\partial^2 \vec{B}}{\partial t^2}$$
$$\Rightarrow v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$$

where

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

is the index of refraction of the substance. For most material,

$$\chi_m \to 0$$

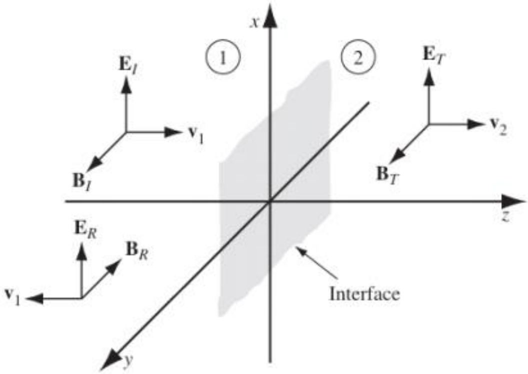
so, we have

$$n = \sqrt{\frac{\mu_0(1+\chi_m)\epsilon}{\mu_0\epsilon_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\kappa} > 1$$

Thus, we conclude that light travels more slowly through matter.

B. REFLECTION AND TRANSMISSION AT NORMAL INCIDENCE

(1) Suppose the xy plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface from the left:



$$\vec{E}_i(z,t) = E_{0i}e^{i(k_1z-\omega t)}\hat{x}$$

$$\vec{B}_i(z,t) = E_{0i}e^{i(k_1z-\omega t)}\frac{1}{v_1}\hat{v}_1 \times \hat{x} = \frac{E_{0i}}{v_1}e^{i(k_1z-\omega t)}\hat{y}$$

It gives rise to a reflected wave and a transmitted wave,

$$\vec{E}_r(z,t) = E_{0r}e^{i(-k_1z-\omega t)}\hat{x}$$

$$\vec{B}_r(z,t) = E_{0r}e^{i(-k_1z-\omega t)}\frac{1}{v_1}(-\hat{v}_1) \times \hat{x} = -\frac{E_{0r}}{v_1}e^{i(-k_1z-\omega t)}\hat{y}$$

$$\vec{E}_t(z,t) = E_{0t}e^{i(k_2z-\omega t)}\hat{x}$$

$$\vec{B}_t(z,t) = E_{0t}e^{i(k_2z-\omega t)}\frac{1}{v_2}\hat{v}_2 \times \hat{x} = \frac{E_{0t}}{v_2}e^{i(k_2z-\omega t)}\hat{y}$$

(2) At $z = 0$, the boundary conditions give

$$\epsilon_1 E_{1\perp} - \epsilon_2 E_{2\perp} = 0$$

$$B_{1\perp} - B_{2\perp} = 0$$

$$E_{1\parallel} - E_{2\parallel} = 0 \Rightarrow E_{0i} + E_{0r} = E_{0t}$$

$$\frac{B_{1\parallel}}{\mu_1}-\frac{B_{2\parallel}}{\mu_2}=0\Rightarrow\frac{1}{\mu_1}\left(\frac{E_{0i}}{v_1}-\frac{E_{0r}}{v_1}\right)=\frac{1}{\mu_2}\frac{E_{0t}}{v_2}\Rightarrow E_{0i}-E_{0r}=\beta E_{0t}$$

where

$$\beta=\frac{\mu_1 v_1}{\mu_2 v_2}\approx\frac{v_1}{v_2}=\frac{n_2}{n_1}$$

Thus, we obtain

$$E_{0r}=\frac{1-\beta}{1+\beta}E_{0i}$$

$$E_{0t}=\frac{2}{1+\beta}E_{0i}$$

(3) The reflection coefficient R and the transmission coefficient T

Since

$$I=\frac{1}{2}\epsilon v E_0^2$$

we have

$$R=\frac{I_r}{I_i}=\left(\frac{E_{0r}}{E_{0i}}\right)^2=\left(\frac{1-\beta}{1+\beta}\right)^2=\left(\frac{1-\frac{n_2}{n_1}}{1+\frac{n_2}{n_1}}\right)^2=\left(\frac{n_1-n_2}{n_1+n_2}\right)^2$$

$$T=\frac{I_t}{I_i}=\frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0t}}{E_{0i}}\right)^2=\left(\frac{2}{1+\beta}\right)^2=\left(\frac{2}{1+\frac{n_2}{n_1}}\right)^2=\frac{4n_1 n_2}{\left(n_1+n_2\right)^2}$$

12-3 Electromagnetic Waves in Conductors

A. ELECTROMAGNETIC WAVES IN CONDUCTOR

(1) Inside a conductor, according to Ohm's law, the (free) current density in a conductor is proportional to the electric field,

$$\vec{J}_f = \sigma \vec{E}$$

Maxwell's equations for linear media is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

$$\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} \cdots\cdots \text{Gauss's law}$$

$$\nabla \cdot \vec{B} = 0$$

(2) The continuity equation for free charge is

$$\nabla \cdot \vec{J}_f + \frac{\partial \rho_f}{\partial t} = 0$$

together with Ohm's law and Gauss's law, gives

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot \vec{J}_f = -\nabla \cdot \sigma\vec{E} = -\frac{\sigma}{\epsilon}\rho_f$$

$$\Rightarrow \rho_f(t) = e^{-(\sigma/\epsilon)t}\rho_f(0)$$

Thus, any initial free charge $\rho_f(0)$ dissipates in a characteristic time $\tau \equiv \epsilon/\sigma$.

(3) As accumulated free charge disappears, from then on, $\rho_f = 0$, we have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$

$$\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0 \cdots\cdots \text{Gauss's law}$$

$$\nabla \cdot \vec{B} = 0$$

Applying the curl, we obtain modified wave equations

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \underbrace{\left(\nabla \cdot \vec{E}\right)}_{=0} - \nabla^2 \vec{E} = -\frac{\partial \left(\nabla \times \vec{B}\right)}{\partial t} = -\mu\sigma\frac{\partial \vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times \left(\nabla \times \vec{B}\right) = \nabla \underbrace{\left(\nabla \cdot \vec{B}\right)}_{=0} - \nabla^2 \vec{B} = \mu\sigma \left(\nabla \times \vec{E}\right) + \mu\epsilon \frac{\partial \left(\nabla \times \vec{E}\right)}{\partial t}$$

$$\Rightarrow \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Assume that

$$\vec{E}(z,t) = E_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{x}$$

$$\vec{B}(z,t) = \frac{\tilde{k}}{\omega} E_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{y}$$

we found

$$\tilde{k} = k + i\kappa$$

$$k = \omega \sqrt{\frac{\mu\epsilon}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right)^{1/2}$$

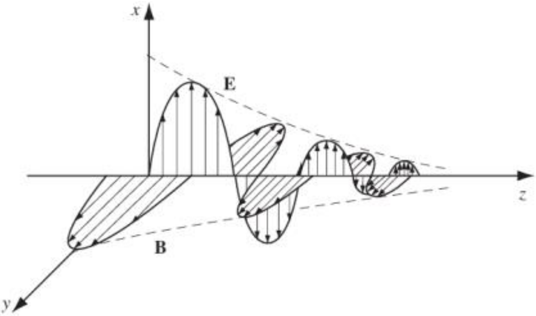
$$\kappa = \omega \sqrt{\frac{\mu\epsilon}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right)^{1/2}$$

$$\left|\tilde{k}\right| = \sqrt{k^2 + \kappa^2} = \omega \left(\mu\epsilon \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right)^{1/2}$$

$$\tan\phi = \frac{\kappa}{k}$$

$$\frac{B_0}{E_0} = \frac{\left|\tilde{k}\right|}{\omega} = \left(\mu\epsilon \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \right)^{1/2}$$

The electric and magnetic fields are



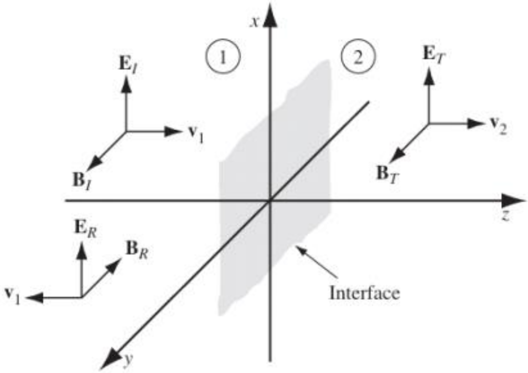
$$\vec{E}(z,t) = E_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{x}$$

$$\vec{B}(z,t) = \frac{|\tilde{k}|}{\omega} E_0 e^{-\kappa z} e^{i(kz-\omega t+\phi)} \hat{y}$$

where $1/\kappa$ is called the skin depth.

B. REFLECTION AT CONDUCTING SURFACE

(1) Suppose the xy plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface from the left:



$$\vec{E}_i(z,t) = E_{0i} e^{i(k_1 z-\omega t)} \hat{x}$$

$$\vec{B}_i(z,t) = \frac{E_{0i}}{v_1} e^{i(k_1 z-\omega t)} \hat{y}$$

It gives rise to a reflected wave and a transmitted wave,

$$\vec{E}_r(z,t) = E_{0r} e^{i(-k_1 z-\omega t)} \hat{x}$$

$$\vec{B}_r(z,t) = -\frac{E_{0r}}{v_1} e^{i(-k_1 z-\omega t)} \hat{y}$$

$$\vec{E}_t(z,t) = E_{0t} e^{i(\tilde{k}_2 z-\omega t)} \hat{x}$$

$$\vec{B}_t(z,t) = \frac{\tilde{k}_2}{\omega} E_{0t} e^{i(\tilde{k}_2 z-\omega t)} \hat{y}$$

(2) At $z = 0$, the boundary conditions give

$$\epsilon_1 E_{1\perp} - \epsilon_2 E_{2\perp} = \sigma_f$$

$$B_{1\perp} - B_{2\perp} = 0$$

$$E_{1\parallel} - E_{2\parallel} = 0$$

$$\frac{B_{1\parallel}}{\mu_1} - \frac{B_{2\parallel}}{\mu_2} = \vec{K}_f \times \hat{n}$$

Since $E_\perp = 0$ on both sides, it gives $\sigma_f = 0$. $B_\perp = 0$

Assume $\vec{K}_f = 0$, we have

$$E_{1\parallel} - E_{2\parallel} = 0 \Rightarrow E_{0i} + E_{0r} = E_{0t}$$

$$\frac{B_{1\parallel}}{\mu_1} - \frac{B_{2\parallel}}{\mu_2} = 0 \Rightarrow \frac{1}{\mu_1}\left(\frac{E_{0i}}{v_1} - \frac{E_{0r}}{v_1}\right) = \frac{\tilde{k}_2}{\mu_2}\frac{E_{0t}}{\omega} \Rightarrow E_{0i} - E_{0r} = \tilde{\beta} E_{0t}$$

where

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{k}_2$$

Thus, we obtain

$$E_{0r} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}} E_{0i}$$

$$E_{0t} = \frac{2}{1+\tilde{\beta}} E_{0i}$$